

MATH PART-II (PAPER-III, Real Analysis)

④ CHAPTER! - Rolle's and Lagrange's Mean-value Theorem

① State and Prove Rolle's theorem

Statement:- If $f(x)$ be a real valued function which is

(i) Continuous over $[a, b]$

(ii) differentiable over $]a, b[$ and

(iii) $f(a) = f(b)$

then there exists a point c in $]a, b[$ such that $f'(c) = 0$

Proof:- Two cases may arise.

In first case if $f(x)$ is a constant function

let $f(x) = k$, a constant (say)

Then conditions ①, ② and ③ are satisfied

Also $f'(c) = 0$ for all x in $]a, b[$

Hence the theorem is valid in this case.

In 2nd Case if $f(x)$ is a variable function

Since $f(x)$ is continuous over a closed interval $[a, b]$, $f(x)$ will attain its (absolute) maximum as well as minimum over $[a, b]$.

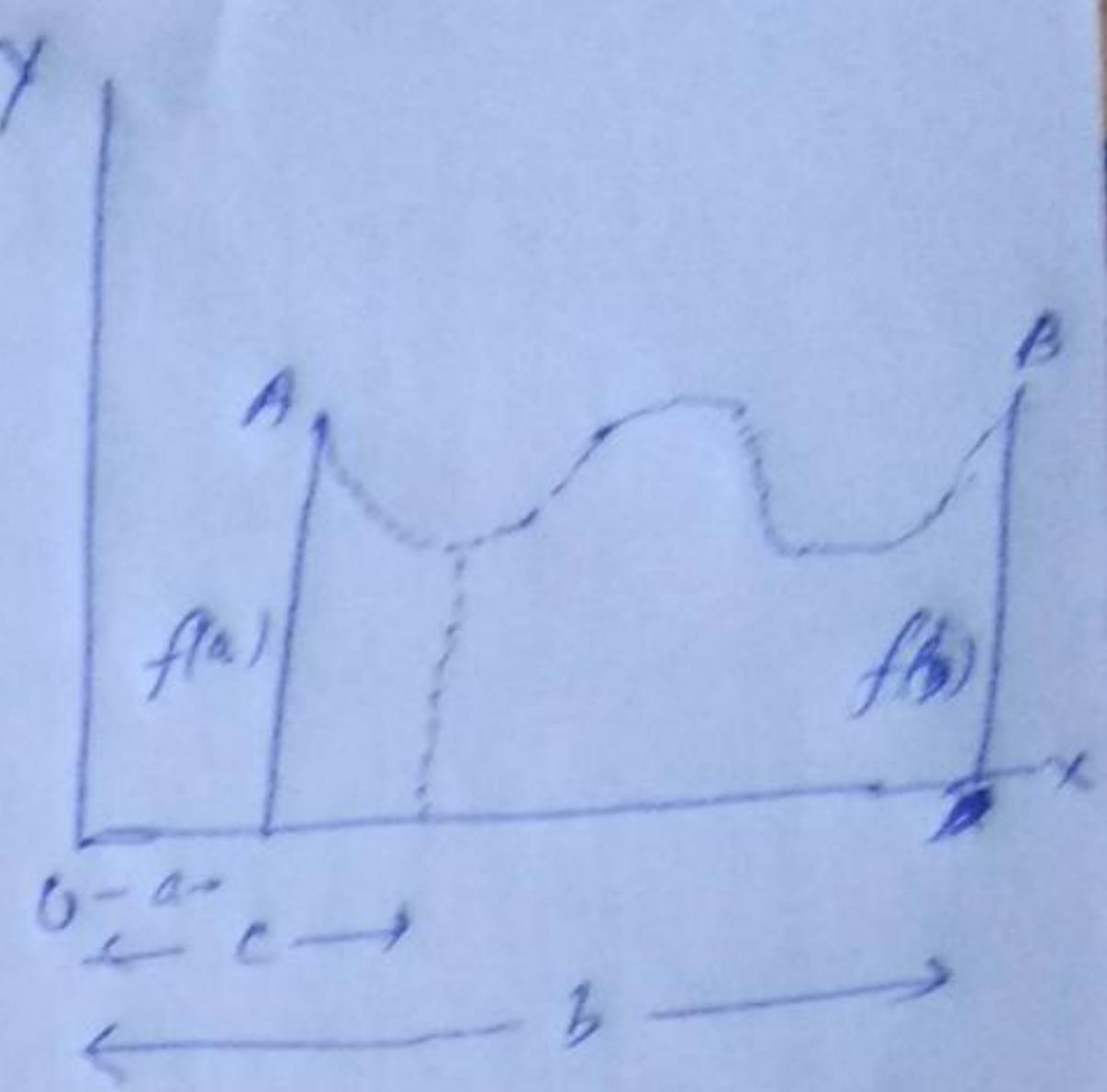
But since $f(x)$ is not a constant function and $f(a) = f(b)$, either the maximum or the minimum will be attained at a point c , $]a, b[$. Now since there is an extremum at c which is not an end point of the interval

$[a, b]$, we have $f'(c) = 0$.

Geometric Significance of Rolle's theorem :-

Rolle's theorem says that

if a function has a continuous graph on the interval $[a, b]$ and the graph has a tangent at every point of (a, b) , then the graph of the function must have a tangent parallel to the x -axis, at least at one point between a and b as shown in the graph.



② State and Prove Lagrange's Mean-Value theorem

Statement :- If a real valued function $f(x)$ defined on $[a, b]$ is (i) Continuous on $[a, b]$

(ii) differentiable on (a, b)

then there exists at least one point $c \in [a, b]$ such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

Proof :- Let us consider a function ϕ defined by $\phi(x) = f(x) + Ax$.

where A is a constant to be determined such that $\phi(a) = \phi(b)$

This gives $f(a) + Aa = f(b) + Ab$

$$\text{or, } \frac{f(b) - f(a)}{b-a} = -A \quad \text{--- (1)}$$

Now the function $\phi(x)$ is the sum of two functions $f(x)$ and Ax both of which are continuous on $[a, b]$ and differentiable on (a, b) . Hence $\phi(x)$ is continuous on $[a, b]$, differentiable on (a, b) and $\phi(a) = \phi(b)$.

So by Rolle's theorem there exist a real number $c \in [a, b]$ such that $\phi'(c) = 0$

$$\text{But } \phi'(x) = f'(x) + A$$

$$\therefore 0 = \phi'(c) = f'(c) + A$$

$$\therefore f'(c) = -A = \frac{f(b) - f(a)}{b-a}, \text{ by (1)}$$

(ii) If in the statement of the theorem, b is replaced by $a+th$, then the number $c \in [a, b]$ may be written as $c = a + \theta h$, for some number θ such that $0 < \theta < 1$

$$\text{Thus } \frac{f(a+th) - f(a)}{th} = f'(a+th)$$

$$\text{or, } f(a+th) - f(a) + th f'(a+\theta h), \text{ where } 0 < \theta < 1$$

Hence the Lagrange's Mean Value theorem may also be stated as follows,

If a real valued function f defined on $[a, a+th]$ is

(i) Continuous on $[a, a+h]$

(ii) differentiable on $]a, a+h[$

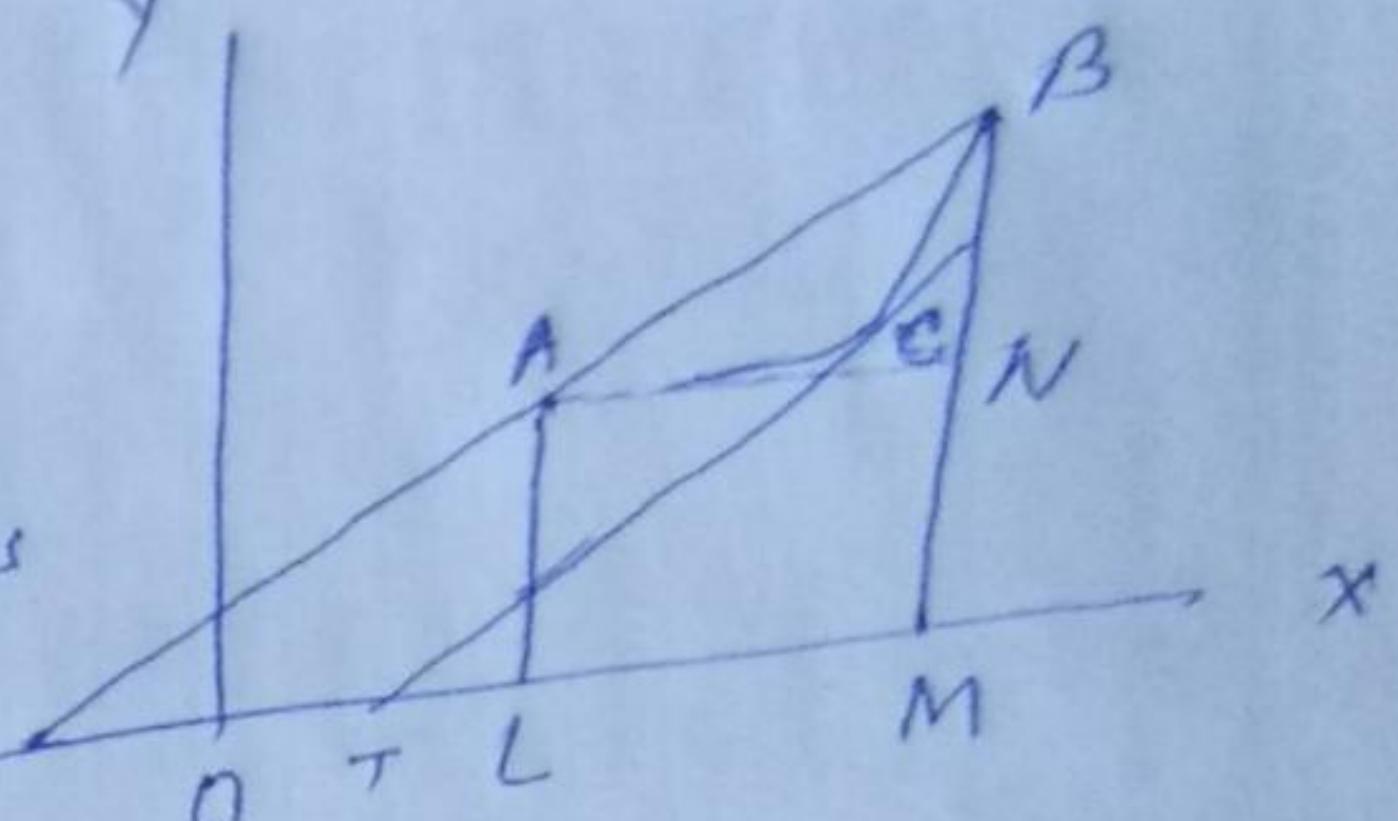
then there exist a number θ , between 0 and 1 such that

$$f(a+h) - f(a) = h f'(a+\theta h), \quad \theta \in]0, 1[$$

Geometrical Interpretation of Lagrange's

Mean Value Theorem:-

let ACB be the graph of $f(u)$ in $[a, b]$ and let a, c, b be the x -coordinates of the points A, C, B



on the Curve $y = f(u)$ such that the relation

$$f(b) - f(a) = b - a \quad f'(c) \text{ is satisfied}$$

Draw AB , BM perpendicular to OX and AN perpendicular to BM . Then $AB = f(a)$, $BM = f(b)$

Let CT be the tangent at C . Then

$$\frac{f(b) - f(a)}{b - a} = \frac{BM - AL}{LM} = \frac{BN}{AN} = \tan \angle BAN$$

Since $f'(c) = \tan \angle CTX$, it follows from Mean Value theorem that $\tan \angle BAN = \tan \angle CTX$

$$\therefore \angle BAN = \angle CTX$$

$\therefore AB$ is parallel to CT .

Hence we have the geometrical interpretation of Lagrange's Mean Value theorem.

① State and Prove Lagrange's Mean Value Theorem

Statement:- If a function f is ① continuous on a closed interval $[a, b]$ and ② differentiable on the open interval (a, b) , then there exists at least one point $c \in [a, b]$ such that

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

Proof:- Consider the function ϕ defined by

$$\phi(x) = f(x) + Ax \quad \text{--- } ①$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$ $\text{--- } ②$

Now, $\phi(a) = f(a) + Aa$ and $\phi(b) = f(b) + Ab$

Hence ② gives

$$f(a) + Aa = f(b) + Ab$$

$$\Rightarrow A(a-b) = f(b) - f(a)$$

$$\Rightarrow A = -\frac{f(b) - f(a)}{b-a} \quad \text{--- } ③$$

Now it is given that $f(x)$ is continuous in the closed interval $[a, b]$ and also x is continuous in the closed interval $[a, b]$.

Hence the function $\phi(x) = f(x) + Ax$ is continuous in the closed interval $[a, b]$.

Again, it is given that $f(x)$ is differentiable in $[a, b]$ and x is differentiable in any interval therefore $\phi(x) = f(x) + Ax$ is also differentiable in $[a, b]$.

Hence all the conditions of Rolle's theorem are satisfied for the function $\phi(x)$.

Therefore, there exists at least one point c of the open interval $[a, b]$ such that $\phi'(c)=0$.

But from the given function $\phi(x)$ as given in ①, we get,

$$\phi'(x) = f'(x) + A$$

$$\Rightarrow \phi'(c) = f'(c) + A, \text{ putting } x=c$$

$$\Rightarrow 0 = f'(c) + A, \quad \therefore \phi'(c) = 0$$

$$\therefore f'(c) = -A \quad \text{--- ④}$$

Now, from ③ and ④, we have

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

which proves the theorem

